Extension of the Homotopy Perturbation Method for Solving Nonlinear Differential-Difference Equations

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In this paper, we have extended the homotopy perturbation method (HPM) to find approximate analytical solutions for some nonlinear differential-difference equations (NDDEs). The discretized modified Korteweg-de Vries (mKdV) lattice equation and the discretized nonlinear Schrödinger equation are taken as examples to demonstrate the validity and the great potential of the HPM in solving such NDDEs. Comparisons are made between the results of the presented method and exact solutions. The obtained results reveal that the HPM is a very effective and convenient tool for solving such kind of equations.

Key words: Homotopy Perturbation Method; Discretized mKdV Lattice Equation; Discretized Nonlinear Schrödinger Equation.

1. Introduction

Many interesting physical phenomena, such as ladder type electric circuit, vibration of particles, collapse of langmuir waves in plasma physics (see [1] and references therein), molecular crystals [2], biophysical systems [3], electrical lattices [4], and, recently, in arrays of coupled nonlinear optical wave guides [5, 6], to cite a few, can be modelled by nonlinear differential-difference equations (NDDEs). Recently, differential-difference equations became a very attracted topic because of the increasing development in nanotechnology fields. Usually, we can use the differential equations to describe various physical problems, but when time or space becomes discontinuous, the differential model becomes invalid. Unlike difference equations which are fully discretized, differential-difference equations are semi-discretized, with some (or all) of their spatial variables discretized, while the time variable is usually kept continuous, see for example [7, 8]. In this paper we will be mainly concerned with outlining an effective procedure that allows us to implement the homotopy perturbation method (HPM) for solving initial value problems of NDDEs. For illustration, we will apply the presented method to the discretized mKdV lattice equation and the discretized nonlinear Schrödinger equation. Recently, Yildirim [9] has applied the HPM to the discrete Korteweg-de Vries (KdV) equation in order to obtain exact solutions.

In recent years a lot of attention has been drawn to solve differential-difference equations using new developed analytical methods, e. g., the exp-function method, the variational iteration method and the parameterized perturbation method, see for example [10 – 13].

Resently, a lot of attention has been drawn to study of the homotopy perturbation method (HPM) to investigate various scientific models. The HPM, based on series approximation, is one among the newly developed analytical methods for strongly nonlinear problems and has been proven successful in solving a wide class of differential equations [14 – 20]. The present method is useful for obtaining both closed form explicit solutions and numerical approximations for both linear and nonlinear differential equations and differential-difference equations as well as we will see in the present paper, and it is of great interest to applied science, engineering, physics, biology, etc.

2. Basic Idea of the Homotopy Perturbation Method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation [14]:

$$L(u)+N(u)-f(r)=0, \quad r\in\Omega, \tag{1}$$

with the boundary conditions

$$B(u,\partial u/\partial n)=0, \quad r\in\Gamma, \tag{2}$$

where L and N are linear and nonlinear differential operators, respectively, B is a boundary operator, $f(r)$ a known analytical function, and Γ is the boundary of the domain Ω.

By the homotopy technique, we construct a homotopy $V(r;p):\Omega\times[0,1]\to\mathbb{R}$ which satisfies

$$\begin{aligned} H(V,p)=(1-p)[L(V)-L(u_0)]& \\ +p[A(V)-f(r)]=0,& \\ p\in[0,1],\ r\in\Omega,& \end{aligned} \tag{3}$$

where $A(V)=L(V)+N(V)$, $p\in[0,1]$ is an embedding parameter, u_0 is an initial approximation of (1), which satisfies the boundary conditions. In most cases $L(u_0)$ is equal to zero.

Obviously, from (3) we will have

$$H(V,0)=L(V)-L(u_0)=0, \tag{4}$$

$$H(V,1)=L(V)+N(V)-f(r)=0. \tag{5}$$

According to the HPM, we can first use the embedding parameter p as a 'small parameter', and assume that the solution of (3) can be written as a power series in p:

$$V=V_0+pV_1+p^2V_2+\dots. \tag{6}$$

Setting $p=1$ results in the approximate solution of (1):

$$u=\lim_{p\to 1}V=V_0+V_1+V_2+\dots. \tag{7}$$

The series in (7) is convergent for most cases, and also the rate of convergent depends on the nonlinear operator $A(V)$ [14].

Let's denote the $m+1$-term approximate solution φ_m by

$$u*\varphi_m=\sum_{k=0}^{m}V_k(r). \tag{8}$$

Theorem (Sufficient condition for convergence). Suppose that X and Y be Banach spaces and N: $X\to Y$ is a contraction nonlinear mapping, that is

$$\forall\, v,v\in X;\ \|N(v)-N(v)\|\le\gamma\|v-v\|, \quad 0<\gamma<1,$$

which according to Banach's fixed point theorem, has the fixed point u, that is $N(u)=u$. The sequence generated by the homotopy perturbation method will be regarded as

$$V_n=N(V_{n-1}), \quad V_{n-1}=\sum_{i=0}^{n-1}u_i, \quad n=1,2,3,\cdots,$$

and suppose that $V_0=v_0=u_0\in B_r(u)$, where $B_r(u)=\{u^*\in X|\|u^*-u\|<r\}$, then we have the following statements:

(i) $\|V_n-u\|\le\gamma^n\|v_0-u\|$,

(ii) $V_n\in B_r(u)$,

(iii) $\lim_{n\to\infty}V_n=u$.

Proof. (i) By the induction method on n, for $n=1$ we have

$$\|V_1-u\|=\|N(V_0)-N(u)\|\le\gamma\|v_0-u\|.$$

Assume that $\|V_{n-1}-u\|\le\gamma^{n-1}\|v_0-u\|$ as an induction hypothesis, then

$$\begin{aligned}\|V_n-u\|&=\|N(V_{n-1})-N(u)\|\le\gamma\|V_{n-1}-u\|\\ &\le\gamma\gamma^{n-1}\|v_0-u\|=\gamma^n\|v_0-u\|.\end{aligned}$$

(ii) Using (i), we have

$$\|V_n-u\|\le\gamma^n\|v_0-u\|\le\gamma^n r<r\Rightarrow V_n\in B_r(u).$$

(iii) Because of $\|V_n-u\|\le\gamma^n\|v_0-u\|$ and $\lim_{n\to\infty}\gamma^n=0$, we drive $\lim_{n\to\infty}\|V_n-u\|=0$, that is, $\lim_{n\to\infty}V_n=u$.

3. Applications

3.1. The Discretized mKdV Lattice Equation

Consider the discretized mKdV lattice equation

$$\frac{\partial u_n}{\partial t}=(1-u_n^2)(u_{n+1}-u_{n-1}) \tag{9}$$

with the initial condition

$$u_n(0)=A\tanh(kn), \tag{10}$$

where k is an arbitrary constant and

$$A=\tanh(k).$$

The exact solution of the problem was given by Wu et al. [21] as

$$u_n(t)=A\tanh(kn+2At). \tag{11}$$

According to (3), a homotopy $V(n,t;p):\Omega\times[0,1]\to$

$\mathbb{R}$ can be constructed as

$$(1-p)[V_t(n,t)-u_t(n,0)]+p\Big\{V_t(n,t) \\ -[1-V^2(n,t)][V(n+1,t)-V(n-1,t)]\Big\}=0. \quad (12)$$

Then (12) can be written as

$$V_t(n,t)-p[1-V^2(n,t)][V(n+1,t) \\ -V(n-1,t)]=0. \quad (13)$$

One can now try to obtain solutions of $V(n-1,t)$, $V(n,t)$, and $V(n+1,t)$ as,

$$V(n+i,t)=\sum_{k=0}^{\infty}p^kV_k(n+i,t),\ i=-1,0,\text{ and } 1. \quad (14)$$

Substituting (14) into (13), and comparing coefficients of terms with identical powers of p, yield

$$\begin{aligned}
p^0 &: \frac{\partial}{\partial t}V_0(n,t)=0,\\
p^1 &: \frac{\partial}{\partial t}V_1(n,t)-(1-(V_0(n,t))^2)\\
&\quad\cdot(V_0(n+1,t)-V_0(n-1,t))=0,\\
p^2 &: \frac{\partial}{\partial t}V_2(n,t)-(1-(V_0(n,t))^2)\\
&\quad\cdot(V_1(n+1,t)-V_1(n-1,t))\\
&\quad+2V_0(n,t)V_1(n,t)\\
&\quad\cdot(V_0(n+1,t)-V_0(n-1,t))=0,\\
p^3 &: \frac{\partial}{\partial t}V_3(n,t)-(1-(V_0(n,t))^2)\\
&\quad\cdot(V_2(n+1,t)-V_2(n-1,t))\\
&\quad+2V_0(n,t)V_1(n,t)\\
&\quad\cdot(V_1(n+1,t)-V_1(n-1,t))\\
&\quad+(2V_0(n,t)V_2(n,t)+(V_1(n,t))^2)\\
&\quad\cdot(V_0(n+1,t)-V_0(n-1,t))=0,\\
&\vdots
\end{aligned} \quad (15)$$

with the following initial conditions:

$$V_i(n,0)=\begin{cases}A\tanh(kn), & i=0,\\ 0, & i=1,2,3,\dots\end{cases} \quad (16)$$

Solving the system (15) with the conditions (16) and using any symbolic computation program yield the following 3-term approximate solution φ_2:

$$\begin{aligned}
u_n(t)*\varphi_2 &= A\tanh(kn)+A(\tanh(kn+k)\\
&-\tanh(kn-k))(1-A^2\tanh^2(kn))t\\
&+\frac{A}{2}[A^2\tanh^2(kn)-1]\\
&\cdot\Big[A^2\tanh(kn)\tanh^2(k(n+1))\\
&-4A^2\tanh(kn)\tanh(k(n-1))\tanh(k(n+1))\\
&+A^2\tanh(kn)\tanh^2(k(n-1))\\
&+A^2\tanh(k(n-2))\tanh^2(k(n-1))\\
&+A^2\tanh(k(n+2))\tanh^2(k(n+1))\\
&+2\tanh(kn)-\tanh(k(n+2)-\tanh(k(n-2))\Big]t^2.
\end{aligned} \quad (17)$$

For this test problem, we continue solving (15), subject to (16), for V_n, $n=0,1,\dots$ until obtaining V_6 and, hence, obtain the approximate solution $\varphi_6=\sum_{j=0}^{6}V_j(n,t)$.

The behaviours of the approximate solutions φ_2 and φ_6 in comparison with the exact solution are illustrated in Figure 1, showing that the more term-approximate solution is the more accurate one for relatively high values of t. This is because the increase in the number of terms used for calculating φ_m will lead to an increase approximate solution radius of convergence.

Moreover, the radius of convergence of the approximate solution can be increased by applying Padé approximants to the obtained series solution φ_m [19].

3.2. The Discretized Nonlinear Schrödinger Equation

Consider the discretized nonlinear Schrödinger equation

$$\mathrm{i}\frac{\partial u_n}{\partial t}=(u_{n+1}+u_{n-1}-2u_n)-|u_n|^2(u_{n+1}+u_{n-1}) \quad (18)$$

with the initial condition

$$u_n(0)=\tanh(k)\mathrm{e}^{\mathrm{i}pn}\tanh(kn), \quad (19)$$

where k and p are arbitrary constants and $\mathrm{i}=\sqrt{-1}$.

The exact solution of the problem was given in [22, 23] as

$$u_n(t)=\tanh(k)\mathrm{e}^{\mathrm{i}[pn+(2-2\cos(p)\,\mathrm{sech}(k))t]} \\ \cdot\tanh(kn+2\sin(p)\tanh(k)t). \quad (20)$$

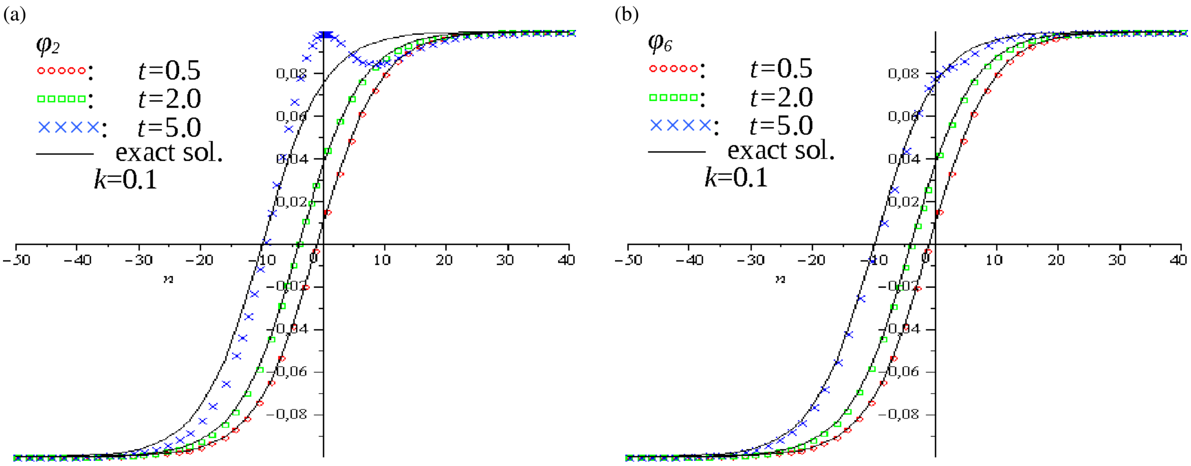

Fig. 1 (colour online). Results for $t = 0.5$, 2, and 5 obtained for the exact solution (solid line) and the HPM solution φ_2, shown in (a), in comparison with φ_6, shown in (b) for the discretized mKdV lattice equation when $k = 0.1$.

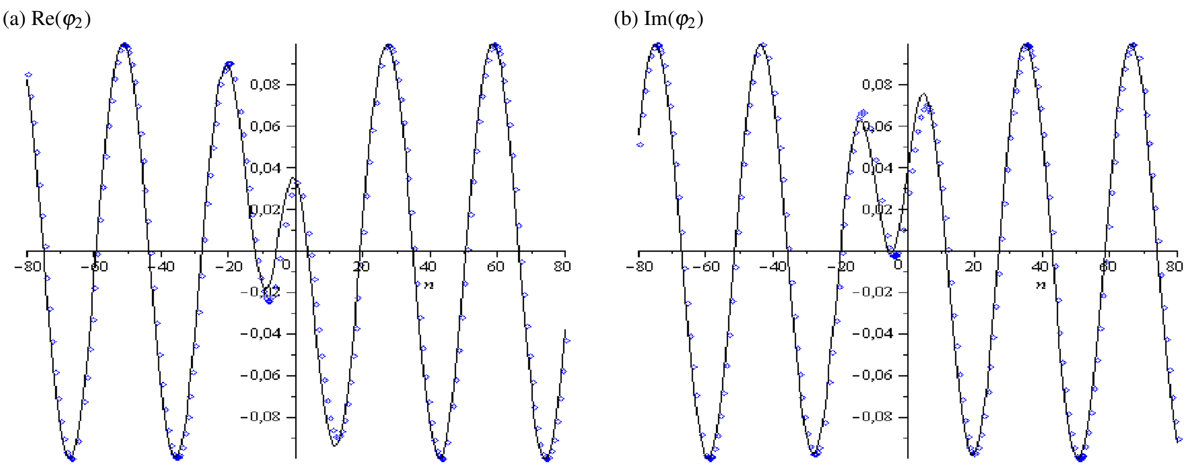

Fig. 2 (colour online). Results for $t = 15$, obtained for the exact solution, real and imaginary parts (solid line) and HPM solution $\mathrm{Re}(\varphi_2)$ shown in (a), and $\mathrm{Im}(\varphi_2)$ shown in (b), for equation (24) when $k = 0.1$ and $p = 0.2$.

As done in [16], we can construct the following homotopy: $V(n,t,p) : \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$\begin{aligned} &V_t(n,t) + \mathrm{i}p[V(n+1,t) + V(n-1,t) - 2V(n,t) \\ &-(V(n,t)\overline{V}(n,t))(V(n+1,t) + V(n-1,t))] = 0, \end{aligned} \tag{21}$$

where $\overline{V}$ is the conjugate of V.

Let us consider the conjugate series solution $\overline{V}$ as

$$\overline{V}(n,t) = \sum_{j=0}^{\infty} p^j \overline{V}_j(n,t). \tag{22}$$

Substituting (14) and (22) into (21) and following the same procedures as done in the previous examples, the

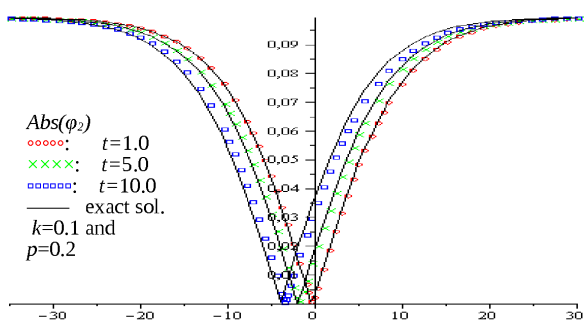

Fig. 3 (colour online). Results for $t = 1$, 5, and 10 obtained for the exact solution, absolute value (solid line) and HPM solution $\mathrm{abs}(\varphi_2)$ for equation (24) when $k = 0.1$ and $p = 0.2$.

3-term approximate solution $\varphi_2 = \sum_{j=0}^{2} V_j(n,t)$ is obtained, and for simplicity we write down only φ_1:

$$\begin{aligned}\varphi_1 &= \tanh(k)\mathrm{e}^{\mathrm{i}pn}\tanh(kn)\\&+\mathrm{i}\tanh(k)\Big[-\mathrm{e}^{\mathrm{i}p(n-1)}\tanh(kn-k)\\&+\tanh^2(k)\tanh^2(kn)\mathrm{e}^{\mathrm{i}p(n+1)}\tanh(kn+k)\\&+\tanh^2(k)\tanh^2(kn)\mathrm{e}^{\mathrm{i}p(n-1)}\tanh(kn-k)\\&-\mathrm{e}^{\mathrm{i}p(n+1)}\tanh(kn+k)+2\mathrm{e}^{\mathrm{i}pn}\tanh(kn)\Big]t.\end{aligned} \tag{23}$$

Some graphical comparisons between the approximate solution φ_2 and the exact solution (20) when $k = 0.1$ and $p = 0.2$ are illustrated in Figures 2 and 3.

From graphical comparisons shown in Figures 2 and 3, it is clear that the 3-term approximate solution φ_2 is an acceptable solution even for relatively high values of t. We can obtain more accurate approximate solutions by solving more equations of the homotopy system and get an approximate solution with a high radius of convergence. Moreover, the HPM approximate solution can be improved by using Padé approximants [19].

4. Conclusions

In this paper, the homotopy perturbation method is extended and utilized to find exact and approximate solutions for the NDDEs, including the discretized mKdV lattice equation and the discretized nonlinear Schrödinger equation. A clear conclusion can be drawn from the obtained results that the considered method is an effective, simple and quite accurate tool for handling nonlinear differential-difference equations in a unified manner. It is predicted that the HPM can be found widely applicable in science and engineering.

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